Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 352 (2009) 57–64

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Semilinear equations on fractal blowups

Ian Schindler [a], Kyril Tintarev [b,*]

[a] *Ceremath–MIP, University of Toulouse 1, F-31400 Toulouse, France*
[b] *Department of Mathematics, Uppsala University, SE-751 06 Uppsala, Sweden*

Received 20 November 2007

Available online 10 January 2008

Submitted by V. Radulescu

Abstract

The paper concerns existence of a ground state for a nonlinear scalar field equation on a blowup fractal, where imbedding of the energy space into L^p is not compact. In absence of invariant transformations involved in conventional concentration-compactness argument, the paper develops convergence reasoning based on the fractal's self-similarity.

Keywords: Fractals; Dirichlet form; Fractal blowup; Semilinear problem; Sobolev-type inequality; Concentration compactness

1. Introduction

Analysis on fractals is a discipline that stems from understanding that many of the physical models need underlying geometric structures that are much more irregular than manifolds, but exhibit infinite iterative properties, stemming from common organizing principles (see Mandelbrot [7]). The central object of this analysis is a counterpart of Laplace–Beltrami operator, called fractal Laplacian, associated with a Dirichlet form $E(u, v)$, called energy, defined initially on some functional space over the fractal Ω (see [5,9,14]). Among the classical equations of mathematical physics, brought into the fractal setting (see the survey of [15]), is the scalar field equation $-\Delta u = f(u)$, previously studied, using variational methods, by Falconer [3], Falconer and Hu [4] and Matzeu [8].

In this paper we consider the scalar field equation defined on fractal blowups (the notion introduced in [13], see also the studies of spectra of the blowup fractal Laplacians in Teplyaev [16] and Sabot [12]), a non-compact medium where the standard variational existence proof, based on compactness of imbedding of the energy space into L^p, cannot be used. Neither, by analogy with the Euclidean case, it is expected that every blowup fractal supports a ground state. Existence proofs of ground state in [3,4] and Matzeu [8] are analogous to the argument for the Euclidean Laplacian. In the case of fractal blowups, the analogy is far from immediate, since the concentration-compactness reasoning (Lions [10,11]) for subcritical nonlinearities is anchored in translation invariance, which allows generalization to metric structures (see [1], also a similar result for manifolds [17]) provided that the metric structure is co-compact with respect to its isometry group. No such isometry group is expected for fractal blowups.

* Corresponding author.
E-mail addresses: ian.schindler@univ-tlse1.fr (I. Schindler), kyril.tintarev@math.uu.se (K. Tintarev).

doi:10.1016/j.jmaa.2007.12.076

There is generally an uncountable family of different blowups of the same fractal, parametrized by the infinite words of indices that determine the sequence of expansion maps. In the present paper we verify that the ground state energy on a fractal blowup is independent of the blowup sequence (Proposition 5.1). A well-known example is the equality of the ground state energy for the scalar field equation on $\mathbb{R}^N$ and on a half-space (both can be identified as dyadic blowups of the fractal $[0,1]^N$). Moreover, $\mathbb{R}^N$ is the only blowup of $[0,1]^N$ (relative to the constituent maps of $[0,1]^N$ as a fractal; definition of the blow-up is found in Section 3 below) that supports a ground state, and consequently it is natural to suspect that in the general case such "stable" blowups are exceptional as well.

This motivates the main statement of the present paper, Theorem 5.2, that for a given underlying fractal there is a blowup that admits a ground state. Similarly to $\mathbb{R}^N$, such ground state can be used to produce a divergent minimizing sequence for any other blowup of the same fractal. It remains an open problem to characterize the "stable" blowups. In particular, is it true that, like in $\mathbb{R}^N$, the blowup sequences that cycle all constituent maps admit a ground state, while the blowups with boundary do not?

An outline of convergence reasoning for the ground state: Our convergence argument for minimizing sequences is based on local isomorphisms, denoted below as $\eta_{I,J,M}$, between the finite blowups corresponding to two different index sequences I, J of length M. These local isometries are compositions of two maps, a "zoom-in" composition of M constituent maps of the fractal and a "zoom-out" composition of the first M members of the blow-up sequence. Since in general the scaling factors for the measure and for the energy on a fractal, associated with different constituent maps, do not have to be the same, the maps $\eta_{I,J,M}$ with the same M but different zoom-in or zoom-out sequences would generally yield different scaling factors, different from 1 (the value that arises when both sequences are the same). In order to prevent this we consider only fractals whose scaling factors are the same for every constituent map.

The paper is organized as follows. In Sections 2 and 3 we repeat definitions and basic properties for self-similar fractals and their blowups, respectively. In Section 4 we prove several preliminary statements, including a Sobolev inequality for blowups. In Section 5 we state and prove Proposition 5.1 and Theorem 5.2.

2. The class of fractals and the energy

We define a class of fractals considered below, a subset of the class of pcf (post-critically finite) fractals, introduced by Kigami [5], as well as correspondent energy spaces following [14]. An essential restriction below is that the constituent maps of the fractal are to have the same scaling factor. The class includes Sierpinski gasket.

Let $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1,\dots,N\}$, be the contractive similitudes, satisfying

$$\left|\psi_i(x) - \psi_i(y)\right| \leqslant \alpha^{-1}|x-y| \tag{2.1}$$

with some $\alpha > 1$ and assume that there is an open set $U \subset \mathbb{R}^N$ such that

$$U \subset \bigcup_i \psi_i(U). \tag{2.2}$$

There exists a unique compact set $\Omega \subset \mathbb{R}^n$ satisfying

$$\Omega = \bigcup_i \psi_i(\Omega) \tag{2.3}$$

and there is a unique Borel regular measure μ on Ω such that for every integrable $u : \Omega \to \mathbb{R}$,

$$\int_\Omega u\,d\mu = \alpha^{-d} \sum_{i=1}^N \int_\Omega u \circ \psi_i\,d\mu \tag{2.4}$$

where $d = \frac{\log N}{\log \alpha}$. The set Ω is called then a self-similar fractal. An equivalent form of (2.4) is

$$\mu(A) = \alpha^{-d} \sum_{i=1}^N \mu\bigl(\psi_i^{-1}(A \cup \Omega)\bigr). \tag{2.5}$$

Let $\partial\Omega$ be a set of fixed points of ψ_k, $k = 1,\dots,N_0$, with some $N_0 \leqslant N$. We assume that Ω is connected and satisfies the finite ramification condition

$$\psi_i\Omega \cap \psi_j\Omega \subset \psi_i\partial\Omega \cap \psi_j\partial\Omega \quad \text{whenever } i \neq j. \tag{2.6}$$

For every function $u:\Omega\to\mathbb{R}$, a positive quadratic form (energy) $E(u)\in[0,+\infty]$, is defined (see [5,14]), satisfying for every $i\in\{1,\dots,N\}$, $u\in H$, $u\circ\psi_i\in H$, and

$$E(u)=\rho\sum_{i=1}^{N}E(u\circ\psi_i) \tag{2.7}$$

with some $\rho>0$. Domain $\mathcal{D}$ of $E(u)$ consists of functions for which $E(u)<\infty$. The Sobolev space $H^1(\Omega)$, which we in what follows abbreviate as H, is defined as the linear space $\mathcal{D}\cap L^2(\Omega)$, equipped with the norm

$$\|u\|^2=E(u)+\|u\|^2_{2,\mu}. \tag{2.8}$$

By definition, H is continuously imbedded into $L^2(\Omega,\mu)$. Moreover, it is compactly imbedded into $L^p(\Omega)$ for all $p\in[1,\infty)$ if $d\leqslant 2$ and for $p\in[1,\frac{2d}{d-2})$ if $d>2$. In what follows we assume that $d<2$, in which case H is also continuously imbedded into $C(\Omega)$. In particular, there exists $C>0$ such that

$$\left(\int_\Omega |u|^p\,d\mu\right)^{\frac{2}{p}}\leqslant C\left(E(u)+\int_\Omega |u|^2\,d\mu\right),\quad u\in H. \tag{2.9}$$

Furthermore, the space H_0 of functions in H vanishing on $\partial\Omega$ is a proper subspace of H. The functions in H admit continuous restrictions to and continuous extensions from the sets $\psi_i\Omega$. The latter are also continuous operators $H_0\to H_0$. As long as it is not ambiguous, we will not distinguish in notations between the functions and their extensions respectively restrictions. In such terms one has, in particular,

$$E\big(u\circ\psi_i^{-1}\circ\psi_j\big)=0\quad\text{whenever } i\neq j. \tag{2.10}$$

3. Self-similar blowups

An infinite blowup Ω^I of Ω, relative to a sequence $I=\{i_1,i_2,\dots\}$, $i_k\in\{1,\dots,N\}$, is the monotone increasing union

$$\bigcup_{M=1}^{\infty}\Omega^I_M,\quad\text{where } \Omega^I_M:=\Phi^I_M\Omega \text{ and } \Phi^I_M:=\psi_{i_1}^{-1}\circ\dots\circ\psi_{i_M}^{-1},\ M\in\mathbb{N}. \tag{3.1}$$

For the sake of consistency we set $\Omega_0=\Omega$ and $\Phi_0=\mathrm{id}$.

The measure μ and the functional E can be extended to Ω^I and to functions thereupon by self-similarity, as follows. The measure μ induces a measure

$$\mu^I_M=\alpha^{dM}\mu\circ{\Phi^I_M}^{-1}\quad\text{on } \Omega^I_M,\ M\in\mathbb{N}. \tag{3.2}$$

From (2.6) and (2.4) easily follows that the measures μ^I_M and μ^I_{M+1} coincide on Ω^I_M, $M=0,1,\dots$. This defines, by $\mu^I_{M+1}|_{\Omega_M}=\mu^I_M$, a measure on a generator set of a σ-algebra on the whole Ω^I, and thus, a Borel measure on Ω^I.

A similar construction yields an energy functional for the blowup. For a finite blowup Ω^I_M we set

$$E^I_M(u)=\rho^{-M}E\big(u\circ\Phi^I_M\big), \tag{3.3}$$

whenever $u\in H^I_M:=\{v\circ{\Phi^I_M}^{-1}, v\in H\}$.

Note that if $u\in H^I_{0,M}:=\{v\circ{\Phi^I_M}^{-1}, v\in H_0\}$ then the extension of u by zero to Ω^I_{M+1} is an element of $H^I_{0,M+1}$ (we will extend the adopted convention not to distinguish in notations between u and its extension to this instance). From (2.7) and (2.10) $E^I_M(u)=E^I_{M+1}(u)$. This defines $E^I(u)$ for any $u\in H^I_0:=\bigcup_{M\in\mathbb{N}}H^I_{0,M}$. The Hilbert space H^I is defined as the completion of H^I_0 with respect to the norm

$$\|u\|_I:=\left(E^I(u)+\int_{\Omega^I}|u|^2\,d\mu^I\right)^{1/2},\quad I\in\{1,\dots,N\}^{\mathbb{N}}.$$

4. Sobolev inequality and a concentration lemma

Let $J = \{j_1, j_2, \ldots\}$, $j_k \in \{1, \ldots, N\}$, $\Phi_M^J = \psi_{j_1}^{-1} \circ \cdots \circ \psi_{j_M}^{-1}$ and let

$$\eta_{I,J,M} \stackrel{\text{def}}{=} \Phi_M^I \circ {\Phi_M^J}^{-1} : \Omega_M^J \to \Omega_M^I. \tag{4.1}$$

Let $I, J \in \{1, \ldots, N\}^{\mathbb{N}}$, $M \in \mathbb{N}$, let

$$\mathcal{J}_M^I \stackrel{\text{def}}{=} \left\{\eta_{I,J,M}\Omega \;\middle|\; J \in \{1, \ldots, N\}^{\mathbb{N}}\right\}$$

and let

$$\mathcal{J}^I \stackrel{\text{def}}{=} \bigcup_{M \in \mathbb{N}} \mathcal{J}_M^I.$$

Lemma 4.1. *Let $I, J \in \{1, \ldots, N\}^{\mathbb{N}}$. The collection of sets $\mathcal{J}^I$ is a covering of Ω^I. Furthermore, for every integrable function w on (Ω^I, μ^I),*

$$\int_{\Omega^I} w\, d\mu^I = \sum_{\eta_{I,J,M}\Omega \in \mathcal{J}^I} \int_{\eta_{I,J,M}\Omega} w\, d\mu^I = \sum_{\eta_{I,J,M}\Omega \in \mathcal{J}^I} \int_{\Omega} w \circ \eta_{I,J,M}\, d\mu \tag{4.2}$$

and for every $u \in H^I$,

$$E^I(u) = \sum_{\eta_{I,J,M}\Omega \in \mathcal{J}^I} E(u \circ \eta_{I,J,M}), \tag{4.3}$$

where the terms in the last two sums, corresponding to J, M respectively J', M' such that $\eta_{I,J,M}|_\Omega = \eta_{I,J',M'}|_\Omega$, are repeated only once.

Proof. Let $x \in \Omega^I$. By definition of Ω^I, there exist $M \in \mathbb{N}$ and $y \in \Omega$, such that $x \in \Phi_M^I y$. Furthermore, by (2.3) there is $i \in \{1, \ldots, N\}$ such that $y \in {\Phi_M^J}^{-1}\Omega$ for some J. This proves that $\mathcal{J}^I$ is a covering.

By density it suffices to prove (4.2) for functions from $H_{0,M}^I$, $M \in \mathbb{N}$, that is, to show that for every μ^I-measurable function w on Ω_M^I,

$$\int_{\Omega_M^I} w\, d\mu_M^I = \sum_{J \in \{1,\ldots,N\}^M} \int_{\eta_{I,J,M}\Omega} w\, d\mu_M^I. \tag{4.4}$$

Let $v = w \circ \Phi_M^I$, then (4.4) is equivalent to

$$\int_{\Omega} v\, d\mu = \sum_{J \in \{1,\ldots,N\}^M} \int_{{\Phi^J}^{-1}\Omega} v\, d\mu = \sum_{J \in \{1,\ldots,N\}^M} \int_{\psi_{j_M} \circ \cdots \circ \psi_{j_1}\Omega} v\, d\mu.$$

The last relation easily follows from (2.3) and (2.6).

It suffices to prove (4.3) for functions in $H_{0,M}^I$, $M \in \mathbb{N}$, that is, to show

$$E_M^I(u) = \sum_{J \in \{1,\ldots,N\}^M} E(u \circ \eta_{I,J,M}) \quad \text{for } u \in H_{0,M}^I. \tag{4.5}$$

If we set $u = w \circ \Phi_M^I$. Then (4.5) is equivalent to

$$E(v) = \rho^{-M} \sum_{J \in \{1,\ldots,N\}^M} E\left(v \circ {\Phi_M^J}^{-1}\right) \quad \text{for } v \in H_0,$$

which in turn is the Mth iteration of (2.7). □

Lemma 4.2. *Let $I, J \in \{1, \dots, N\}^N$ and let $\Omega' \subset \Phi_M^J \Omega$ be a μ^J-measurable set. For any μ^I-measurable function $u : \Omega^I \to \mathbb{R}$,*

$$\int_{\eta_{I,J,M}\Omega'} u\, d\mu^I = \int_{\Omega'} u \circ \eta_{I,J,M}\, d\mu^J. \tag{4.6}$$

Proof. Using (3.2)

$$\int_{\eta_{I,J,M}\Omega'} u\, d\mu^I = \int_{\Phi_M^I \Phi_M^{J\,-1}\Omega'} u\, d\mu_M^I = \alpha^{-Md} \int_{\Phi_M^{J\,-1}\Omega'} u \circ \Phi_M^I\, d\mu = \int_{\Omega}^{\prime} u \circ \Phi_M^I \Phi_M^{J\,-1}\, d\mu_M^J$$

with understanding that the composition $u \circ \eta_{I,J,M}$, although not defined on the whole Ω^I, is defined on the domain of the integration. □

Corollary 4.3. *For any μ^I-measurable function $u : \Omega^I \to \mathbb{R}$,*

$$\int_{\eta_{I,J,M}\Omega} u\, d\mu^I = \int_{\Omega} u \circ \eta_{I,J,M}\, d\mu. \tag{4.7}$$

Lemma 4.4. *Let $I, J \in \{1, \dots, N\}^{\mathbb{N}}$, $M \in \mathbb{N}$. For every $u \in H_{0,M}^I$,*

$$E_M^J(u \circ \eta_{I,J,M}) = E_M^I(u).$$

Proof. By (3.3) and the definition of $\eta_{I,J,M} = \Phi_M^I \circ \Phi_M^{J\,-1}$,

$$E_M^J(u \circ \eta_{I,J,M}) = \rho^{-M} E\big(u \circ \Phi_M^I\big) = E_M^I(u). \qquad \square$$

Proposition 4.5. *Let $p > 2$. The following Sobolev inequality holds true*:

$$\left(\int_{\Omega^I} |u|^p\, d\mu^I\right)^{\frac{2}{p}} \leqslant C\left(E^I(u) + \int_{\Omega^I} |u|^2\, d\mu^I\right), \quad u \in H^I. \tag{4.8}$$

Proof. It suffices to consider $u \in H_0$. From (2.9) for $u \circ \eta_{I,J,M}$ and Corollary 4.3 for $J \in \{1, \dots, N\}^{\mathbb{N}}$, $M \in \mathbb{N}$, follows

$$\left(\int_{\eta_{I,J,M}\Omega} |u|^p\, d\mu^I\right)^{\frac{2}{p}} \leqslant C\left(E(u \circ \eta_{I,J,M}|_\Omega) + \int_{\eta_{I,J,M}\Omega} |u|^2\, d\mu^I\right), \quad u \in H_0^I. \tag{4.9}$$

Add the inequalities above over $J \in \{1, \dots, N\}^M$, use Lemma 4.1 and subadditivity of the left-hand side. □

The following result is analogous to the "vanishing lemma" from [6].

Lemma 4.6. *Let $u_k \in H^I$ be a bounded sequence and assume that for every sequence $\Omega_k \in \mathcal{J}^I$, $\Omega_k = \eta_{I,J_k,M_k}\Omega$, $u_k \circ \eta_{I,J_k,M_k}|_\Omega \to 0$ in $L^p(\Omega, \mu)$, then $u_k \to 0$ in $L^p(\Omega^I, \mu^I)$.*

Proof. From (4.9) it is immediate for all $u \in H^I$ that

$$\int_{\eta_{I,J,M}\Omega} |u|^p\, d\mu^I \leqslant C\left(E(u \circ \eta_{I,J,M}|_\Omega) + \int_{\eta_{I,J,M}\Omega} |u|^2\, d\mu^I\right)\left(\int_{\eta_{I,J,M}\Omega} |u|^p\, d\mu^I\right)^{1-\frac{2}{p}}.$$

Adding the inequalities above for $\eta_{I,J,M})\Omega \in \mathcal{J}^I$ and using Lemma 4.1 together with subadditivity of the left-hand side we obtain, setting $u = u_k$,

$$\int_{\Omega^I} |u_k|^p \, d\mu^I \leqslant C\left(E(u_k) + \int_{\Omega^I} |u_k|^2 \, d\mu^I\right) \sup_{\eta_{I,J,M}^{-1}\Omega \in \mathcal{J}^I} \left(\int_{\Omega} |u_k \circ \eta_{I,J,M}|^p \, d\mu\right)^{1-\frac{2}{p}}.$$

Let $(\Omega_k) \in \mathcal{J}^I$, $\Omega_k = \eta_{I,J_k,M_k}^{-1}\Omega$, be such that

$$\int_{\Omega} |u_k \circ \eta_{I,J_k,M_k}|^p \, d\mu \geqslant \frac{1}{2} \sup_{J,M \in \mathcal{J}_I} \int_{\Omega} |u_k|_{\eta_{I,J,M}^{-1}\Omega} \circ \eta_{I,J,M}|^p \, d\mu.$$

Then, by the assumption of the lemma,

$$\int_{\Omega^I} |u_k|^p \, d\mu^I \leqslant C\left(E(u_k) + \int_{\Omega^I} |u_k|^2 \, d\mu^I\right)\left(\int_{\Omega} |u_k \circ \eta_{I,J_k,M_k}|^p \, d\mu\right)^{1-\frac{2}{p}} \to 0. \qquad \square$$

5. Existence of the minimizers

Proposition 5.1. *Let $p > 2$ and let*

$$c^I = \inf\left\{E^I(u) + \int_{\Omega^I} |u|^2 \, d\mu^I \;\Big|\; u \in H^I, \int_{\Omega^I} |u|^p = 1\right\}. \tag{5.1}$$

Then for every $I, J \in \{1, \dots, N\}^{\mathbb{N}}$, $c^I = c^J$.

Proof. It suffices to show that $c^I \geqslant c^J$. Let $\epsilon > 0$ and let $u_\epsilon \in H_0^I$ be such that $\int_{\Omega^I} |u_\epsilon|^p \, d\mu^I = 1$ and $E^I(u) + \int_{\Omega^I} |u_\epsilon|^2 \, d\mu^I \leqslant c^I + \epsilon$. By definition of H_0^I there exists $M_\epsilon \in \mathbb{N}$ such that $u_\epsilon \in H_{0,M_\epsilon}^I$. Let $v_\epsilon = u_\epsilon \circ \eta_{I,J,M_\epsilon}$. Then by Lemmas 4.4 and 4.2, we have $E^J(v_\epsilon) = E^I(u_\epsilon)$, $\int_{\Omega^J} |v_\epsilon|^2 \, d\mu^J = \int_{\Omega^I} |u_\epsilon|^2 \, d\mu^I$ and $\int_{\Omega^J} |v_\epsilon|^p \, d\mu^J = \int_{\Omega^I} |u_\epsilon|^p \, d\mu^I = 1$. Consequently, $c^J \leqslant c^I + \epsilon$. Since ϵ, I and J are arbitrary, the lemma follows. $\square$

Due to the proposition above we may denote the common value of constants c^I, $I \in \{1, \dots, N\}^{\mathbb{N}}$, as c^Ω. Note that $c^\Omega > 0$ due to (4.9).

Theorem 5.2. *Let Ω be a self-similar fractal equipped with the energy E as defined in Section 2. Let Ω^I, $I \in \{1, \dots, N\}^{\mathbb{N}}$, be its blowup with correspondent energy E^I as defined in Section 3, and let $p > 2$. Then there exists $J \in \{1, \dots, N\}^{\mathbb{N}}$ such that the minimum in (5.1) with $I = J$ is attained.*

Proof. The proof consists of three steps. On the first step one moves an Ω-sized "spotlight" $\eta_{I,J,M}\Omega$ to find a weak limit of the minimizing sequence in restriction to the spotlight domain. At this step we also obtain the multi-index $J \in \{1, \dots, N\}^N$ from the sequence of spotlight shifts $\eta_{I,J,M}$.

On the second step we expand the size of the spotlight to the blowup Ω^J, which is generally different from Ω^I, and which becomes a domain of the weak limit for a shifted sequence of u_k.

The third step is a standard concentration compactness argument based on the Brézis–Lieb lemma for functions on Ω^J.

Step 1. Let $u_k \in H_0^I$ be a minimizing sequence for (5.1), that is, $\int_{\Omega^I} |u_k|^p \, d\mu^I = 1$ and $E^I(u_k) + \int_{\Omega^I} |u_k|^2 \, d\mu^I \to c^\Omega$. Since u_k does not converge to zero in $L^p(\Omega^I, \mu^I)$, by Lemma 4.6, there is a sequence of $J_k \in \{1, \dots, N\}^N$, $M_k \in \mathbb{N}$, such that $u_k \circ \eta_{I,J_k,M_k}$ does not converge in $L^p(\Omega, \mu)$ to zero, and, since the local Sobolev imbedding (2.9) is compact, the sequence does not converge to zero weakly in H. It is bounded, however, in H due to Lemmas 4.1 and 4.4. Thus, there exists $w_1 \in H$, such that on a renumbered subsequence, $u_k \circ \eta_{I,J_k,M_k}|_\Omega \rightharpoonup w_1 \neq 0$ in H.

Step 2. Consider the sequence of maps

$$\eta_{I,J_k,M_k} = \Phi_{M_k}^I \circ \psi_{j_{M_k}^k} \circ \dots \circ \psi_{j_1^k} : \Omega_{M_k}^{J_k} \to \Omega_{M_k}^{I_k}. \tag{5.2}$$

We recall that we consider all functions of the class H_0^I as extended by zero to all of Ω^I. Without loss of generality, as both composition chains $\Phi^I_{M_k}$ and $\Phi^J_{M_k}$ may be lengthened with mutually cancelling terms, we may assume that the values of renamed M_k are so large that $u_k \in H^I_{0,M_k}$. In more detail, assume first that $u_k \in H^I_{0,M_k+m_k}$, with some $m_k \in \mathbb{N}$, set $j_{M_k+m} \overset{\text{def}}{=} i_{M_k+m}$, $m = 1, \dots, m_k$, and let $\Phi_{M_k+m_k} := \psi_{j_{M_k+m_k}} \circ \dots \circ \psi_1$, then $\eta_{I,J_k,M_k} = \Phi^I_{M_k+m_k} \Phi^J_{M_k+m_k}$. The map

$$\eta_{I,J_k,M_k+m_k} : \Omega^{J_k}_{M_k+m_k} \to \Omega^{I_k}_{M_k+m_k}$$

is an extension of the map $\eta_{I,J_k,M_k} : \Omega^{J_k}_{M_k} \to \Omega^{I_k}_{M_k}$. As we rename $M_k + m_k$ as M_k, the map η_{I,J_k,M_k+m_k} acquires the notation η_{I,J_k,M_k} of the map it extended.

There is a renamed subsequence J^1_k where $j_{1,k} \in \{1, \dots, N\}$ is constant, to be denoted as j_1. Moreover if for a given $m \in \mathbb{N}$ there is a subsequence J^m_k where $j_{1,k}, \dots, j_{m,k}$ are constant, then it has an extraction where $j_{m+1,k}$ is constant as well. Let $J := (j_1, j_2, \dots)$. Finally, rename $J^{M_k}_k$ as J_k so that $j_{i,k} = j_i$ for $i = 1, 2, \dots, M_k$ (so that the componentwise limit of J_k is J).

The map η_{I,J_k,M_k} is defined then as a map $\Omega^J_{M_k} \to \Omega^I_{M_k}$ (since the components of J_k with $k > M_k$ are not involved in the definition of η_{I,J_k,M_k}) and $u_k \circ \eta_{I,J_k,M_k} : \Omega^J_{M_k} \to \mathbb{R}$ is a bounded sequence in H^J. Then, on a renamed subsequence, $u_k \circ \eta_{I,J_k,M_k} \rightharpoonup w$ in H^J. Due to Step 1, $w|_\Omega = w_1 \neq 0$.

Step 3. Let $v_k := u_k \circ \eta_{I,J_k,M_k} - w$. By Lemma 4.2 and the Brézis–Lieb lemma (see [2]),

$$1 = \lim \int_{\Omega^I} |u_k|^p \, d\mu^I = \lim \int_{\Omega^J} |u_k \circ \eta_{I,J_k,M_k}|^p \, d\mu^J = \lim \int_{\Omega^J} |v_k|^p \, d\mu^J + \int_{\Omega^J} |w|^p \, d\mu^J. \tag{5.3}$$

We also have, since $v_k \rightharpoonup 0$ in H^J, using the scalar products of, respectively, H^I and H^J,

$$c^\Omega = \lim \|u_k\|^2_I = \lim \|u_k \circ \eta_{I,J_k,M_k}\|^2_J = \lim \|v_k\|^2_J + \|w\|^2_J. \tag{5.4}$$

Let $t := \lim \|v_k\|^p_{p,\mu^J}$, then $\|w\|^p_{p,\mu^J} = 1 - t$, and by (5.1), from (5.4) follows

$$c^\Omega \geqslant c^\Omega t^{p/2} + c^\Omega (1-t)^{p/2},$$

which is true only if $t = 1$ (which is impossible since $w_1 \neq 0$ and thus $w \neq 0$) or $t = 0$. Therefore, $\|w\|^p_{p,\mu^J} = 1$, which easily yields that w is a minimizer for (5.1) with $I = J$. $\square$

Acknowledgments

The authors thank their mathematics departments for reciprocal warm hospitality.

References

[1] M. Biroli, I. Schindler, K. Tintarev, Semilinear equations on Hausdorff spaces with symmetries, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 27 (2003) 175–189.
[2] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[3] K.J. Falconer, Semilinear PDEs on self-similar fractals, Comm. Math. Phys. 206 (1999) 235–245.
[4] K.J. Falconer, Jiaxin Hu, Non-linear elliptical equations on the Sierpinski gasket, J. Math. Anal. Appl. 240 (1999) 552–573.
[5] J. Kigami, Analysis on Fractals, Cambridge Tracts in Math., vol. 143, Cambridge University Press, Cambridge, 2001, 226 pp.
[6] E. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, Invent. Math. 74 (1983) 441–448.
[7] B. Mandelbrot, The Fractal Geometry of Nature, Freeman, San Francisco, 1982.
[8] M. Matzeu, Mountain pass and linking type solutions for semilinear Dirichlet forms, in: Recent Trends in Nonlinear Analysis, in: Progr. Nonlinear Differential Equations Appl., vol. 40, Birkhäuser, 2000, pp. 217–231.
[9] U. Mosco, Dirichlet forms and self-similarity, in: New Directions in Dirichlet Forms, in: AMS/IP Stud. Adv. Math., vol. 8, Amer. Math. Soc., Providence, RI, 1998, pp. 117–155.
[10] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–1453.

[11] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.
[12] C. Sabot, Pure point spectrum for the Laplacian on unbounded nested fractals, J. Funct. Anal. 173 (2000) 497–524.
[13] R. Strichartz, Fractals in the large, Canad. J. Math. 50 (1998) 638–657.
[14] R. Strichartz, Function spaces on fractals, J. Funct. Anal. 198 (2003) 43–83.
[15] R. Strichartz, Differential Equations on Fractals. A Tutorial, Princeton University Press, Princeton, NJ, 2006.
[16] A. Teplyaev, Spectral analysis on infinite Sierpinski gaskets, J. Funct. Anal. 159 (1998) 537–567.
[17] K. Tintarev, K.-H. Fieseler, Concentration Compactness: Functional-Analytic Foundations and Applications, Imperial College Press, 2006, 270 pp.